3.2 Rolle's Theorem and the Mean Value Theorem

Objective: Given the definition of the Rolle's and Mean Value Theorems students will be able to understand and use them in problems.

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#3- 51(*3), 53, 59-60, 64

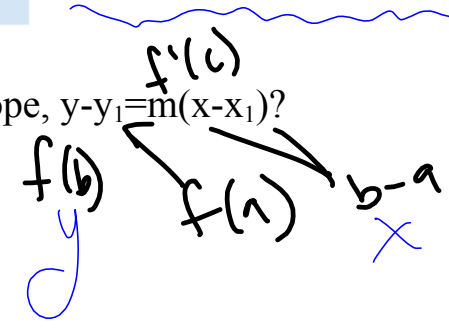
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

A useful alternative form of the Mean Value Theorem is as follows: If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number c in (a, b) such that

$$f(b) = f(a) + (b - a)f'(c).$$

Alternative form of Mean Value Theorem

How is this related to point slope, $y-y_1=m(x-x_1)$?



Example

Given
$f(x) = -x^2 + 5$

a) find the equation of the secant line joining (-1, 4) and (2, 1).
b) Use the mean value theorem to determine a point c in the interval (-1, 2) such that the tangent line at c is parallel to the secant line.
c) Find the equation of the tangent line through c.
d) graph all

a) $m = \frac{4-1}{-1-2} = \frac{3}{-3} = -1$

$y - 4 = -1(x+1)$

$y = -1(x+1)+4$ $y = -1(x-2)+1$

$y = -1x + 3$ same $y = -1x + 3$

b) $-1 = -2x$ $f'(x) = -2x$

$x = 1/2$ $c = 1/2$

c) $(1/2, 4.75)$

$y - 4.75 = -1(x - \frac{1}{2})$

Instantaneous rate of change = average rate of change

Example

Given
$f(x) = x^2 - x - 12$

a) find the equation of the secant line joining (-2, -6) and (4,0).
b) Use the mean value theorem to determine a point c in the interval (-2, 4) such that the tangent line at c is parallel to the secant line.
c) Find the equation of the tangent line through c.
d) graph all

a) $\frac{-6-0}{-2-4} = \frac{-6}{-6} = 1$

$y = x - 4$

b) $f'(x) = -2x - 1$

$1 = -2x - 1$

$x = 1$ $(1, -12)$

c) $y + 12 = (x - 1)$

$y = (x-1) - 12$

Instantaneous rate of change = average rate of change

Physical Interpretation

If we think of the difference quotient $(f(b) - f(a))/(b - a)$ as the average change in f over $[a, b]$ and $f'(c)$ as an instantaneous change, then the Mean Value Theorem says that the instantaneous change at some interior point must equal the average change over the entire interval.

EXAMPLE 4 Interpreting the Mean Value Theorem

If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is $352/8 = 44$ ft/sec, or 30 mph. At some point during the acceleration, the theorem says, the speedometer must read exactly 30 mph (Figure 4.15).

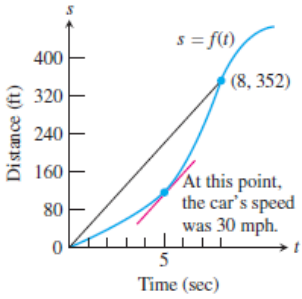
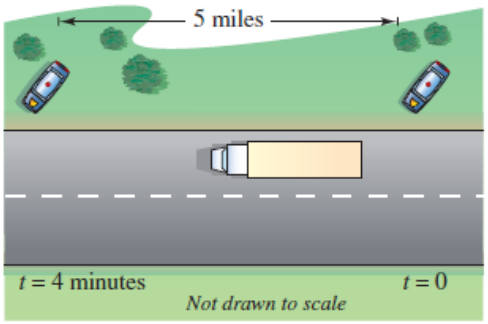


Figure 4.15 (Example 4)

EXAMPLE 4 Finding an Instantaneous Rate of Change

Two stationary patrol cars equipped with radar are 5 miles apart on a highway, as shown in Figure 3.14. As a truck passes the first patrol car, its speed is clocked at 55 miles per hour. Four minutes later, when the truck passes the second patrol car, its speed is clocked at 50 miles per hour. Prove that the truck must have exceeded the speed limit (of 55 miles per hour) at some time during the 4 minutes.



At some time t, the instantaneous velocity is equal to the average velocity over 4 minutes.

Solution Let $t = 0$ be the time (in hours) when the truck passes the first patrol car. The time when the truck passes the second patrol car is

$$t = \frac{4}{60} = \frac{1}{15} \text{ hour.}$$

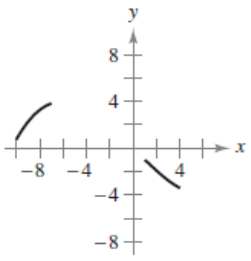
By letting $s(t)$ represent the distance (in miles) traveled by the truck, you have $s(0) = 0$ and $s\left(\frac{1}{15}\right) = 5$. So, the average velocity of the truck over the five-mile stretch of highway is

$$\text{Average velocity} = \frac{s(1/15) - s(0)}{(1/15) - 0} = \frac{5}{1/15} = 75 \text{ miles per hour.}$$

Assuming that the position function is differentiable, you can apply the Mean Value Theorem to conclude that the truck must have been traveling at a rate of 75 miles per hour sometime during the 4 minutes. ■

CAPSTONE

64. ***Graphical Reasoning*** The figure shows two parts of the graph of a continuous differentiable function f on $[-10, 4]$. The derivative f' is also continuous. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



(a) Explain why f must have at least one zero in $[-10, 4]$.

(b) Explain why f' must also have at least one zero in the interval $[-10, 4]$. What are these zeros called?

(c) Make a possible sketch of the function with one zero of f' on the interval $[-10, 4]$.

(d) Make a possible sketch of the function with two zeros of f' on the interval $[-10, 4]$.

(e) Were the conditions of continuity of f and f' necessary to do parts (a) through (d)? Explain.